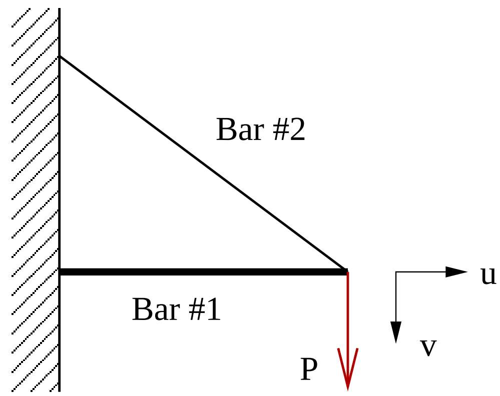
CE 570: Advanced Structural Mechanics

Example of Castigliano's First Theorem

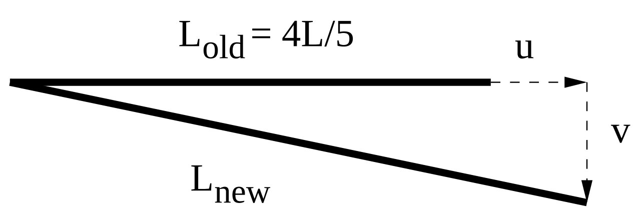


Given: A pair of truss bars are connected as shown and subject to a vertically downward loading *P*. Bar #1 has a length = $\frac{4}{5}L$ and cross-sectional area = *A*, while Bar #2 has a length = *L* and a cross-sectional area = $\frac{1}{2}A$. Both bars are linearly elastic with a modulus of elasticity = *E*.

Required: Find the horizontal and vertical deflection of the joint due to *P*.

Solution: 1. Enforce displacement compatibility – how are *u* and *v* related to the axial deformation of each bar? (Assume that *u* and *v* are both small compared to *L*.)

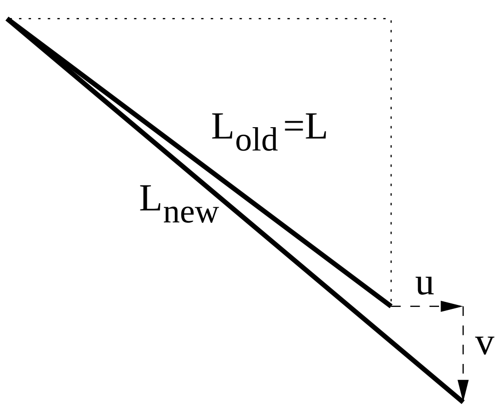
Bar #1: $$L_{new} = \sqrt{\left(\tfrac{4}{5}L + u\right)^2 + v^2} = \tfrac{4}{5}L\sqrt{\left(1 + \tfrac{5u}{4L}\right)^2 + \left(\tfrac{5v}{4L}\right)^2} = \tfrac{4}{5}L\sqrt{1 + 2*\tfrac{5u}{4L} + \left(\tfrac{5u}{4L}\right)^2 + \left(\tfrac{5v}{4L}\right)^2}$$



But, for $\varepsilon << 1$, we have $\sqrt{1+\varepsilon} \approx 1 + \frac{1}{2}\varepsilon + O\left(\varepsilon^2\right)$.

$$\therefore\ L_{new} \approx \tfrac{4}{5}L\left(1 + \tfrac{1}{2} * \tfrac{5u}{4L}\right) = \tfrac{4}{5}L + u \Rightarrow \delta_1 = L_{new} - L_{old} = u$$

Bar #2:



$$\begin{aligned} L_{new} &= \sqrt{\left(\tfrac{4}{5}L + u\right)^2 + \left(\tfrac{3}{5}L + v\right)^2} \\ &= L\sqrt{1 + 2\left(\tfrac{4u}{5L} + \tfrac{3v}{5L}\right) + O\left(u^2, v^2\right)} \\ &\approx L\left\{1 + \tfrac{1}{2} * 2\left(\tfrac{4u}{5L} + \tfrac{3v}{5L}\right)\right\} \end{aligned}$$

$$\therefore\ L_{new} \approx L + \tfrac{4u}{5} + \tfrac{3v}{5} \Rightarrow \delta_2 = L_{new} - L_{old} = \tfrac{4u}{5} + \tfrac{3v}{5}.$$

2. Compute strain energy – for each bar, $U_i = \int_0^{L_i} \frac{P_i(x)^2}{2EA_i(x)} dx = \frac{P_i^2}{2EA_i} \int_0^{L_i} dx = \frac{P_i^2 L_i}{2EA_i}$, with $\delta_i = \frac{P_i L_i}{EA_i}$.

Therefore, $P_i = \frac{EA_i \delta_i}{L_i} \Rightarrow U_i = \left(\frac{EA_i \delta_i}{L_i} \right)^2 \frac{L_i}{2EA_i} = \frac{EA_i \delta_i^2}{2L_i}$.

So, $U_{total} = \sum_{i=1}^{2} U_i = \frac{EA}{2 * \frac{4}{5} L} (u)^2 + \frac{E * \frac{1}{2} A}{2 * L} \left(\frac{4u}{5} + \frac{3v}{5} \right)^2 = \frac{0.785EA}{L} u^2 + \frac{0.24EA}{L} uv + \frac{0.09EA}{L} v^2$.

3. Castigliano's 1st theorem – vertical force is associated with vertical displacement; horizontal force is associated with horizontal displacement:

$$P = \frac{\partial U_{total}}{\partial v} = \frac{0.24EA}{L} u + \frac{0.18EA}{L} v; \; 0 = \frac{\partial U_{total}}{\partial u} = \frac{1.57EA}{L} u + \frac{0.24EA}{L} v.$$

Solve these two equations simultaneously for *u* and *v*:

$$u = -1.0667 \frac{PL}{EA}; \; v = 6.9778 \frac{PL}{EA}.$$ **Answer**